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ITERATION OF NUMBER-THEORETIC FUNCTIONS

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Iteration of number-theoretic functions *)

by

Herman J.J. te Riele

ABSTRACT

This is a concise survey of literature on sequences which arise when a number-theoretic function $f : \mathbb{N} \to \mathbb{N}$ is iteratively applied to a given starting number. Many of the functions f discussed are easy to implement on a programmable calculator.

Functions f for which $f(n) > n$ for all $n \in \mathbb{N}$, or $f(n) < n$ for all $n \in \mathbb{N}$, are excluded from this survey.

KEY WORDS & PHRASES: *Number-theoretic functions, iteration*

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1. INTRODUCTION

Everyone who possesses a scientific calculator may have experimented with it by repeatedly pressing one (or more) buttons. For example, when we start with 0 and repeatedly press the COS button (while the calculator is in degree mode) we obtain
1, 0.999847695, 0.999847742, 0.999847742,... . Apparently, this sequence tends very quickly to a fixed point. Some more experimenting soon reveals that *every* iterative sequence, generated by the COS - function tends to the same fixed point. Very recently, C.H. WAGNER [56] has explained this phenomenon with the *Contraction Mapping Principle* [2]. More important, he showed that many topics as Newton's method, Picard's method for showing existence and uniqueness of solutions to initial value problems, as well as many other iterative processes, are simply different manifestations of this Principle. However, an essential condition for application of this Principle is that the function to be iterated is *continuous*.

The subject of this paper is iteration of functions f which are *not* continuous, but which have as their range and domain the set $\mathbb{N}$ of *positive integers* (or the set $\mathbb{N}^p$ of p-vectors of positive integers). We will call the resulting sequences f-*iterative sequences*. We will *not* consider functions f for which $f(n) > n$ for all $n \in \mathbb{N}$ or $f(n) < n$ for all $n \in \mathbb{N}$.

We present a concise survey of literature on f-iterative sequences for

(i) those who are interested in and working on these sequences, but do not want to duplicate known theoretical or computational results, and

(ii) those who have obtained results and want to know whether these are already known.

f-iterative sequences have been studied in the literature for various f and some of the questions which have been raised, turned out to be extremely difficult, if not "hopeless". Nevertheless, anyone whose calculator is *programmable* can easily experiment with such sequences and discover surprising phenomena. In order to demonstrate this, we give three examples.

EXAMPLE 1.

$$f(n) := \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3 \\ [n\sqrt{3}] & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$$

(By [x] we mean the greatest integer ≤ x.) By means of an HP 41C the following four sequences were generated:

1, 1; 2, 3, 1; 4, 6, 2;
5, 8, 13, 22, 38, 65, 112, 193, 334, 578, 1001, 1733, 3001, 5197, 9001, 15590, 27002, 46768, 81004, 140303, 243011, 420907, 729032, 1262720, 2187095, 3788159, 6561283, 11364475, 19683848, 34093424, *59051542*, 102280271, 177154626, *59051542*.

Since the HP 41C calculates with an accuracy of about 10 decimal digits, the final terms in the last sequence are doubtful. On a TI 59, which calculates with an accuracy of about 13 digits, it appeared that the term 17715462<u>6</u> should be 17715462<u>5</u>. Proceeding with the TI 59, the sequence again increased monotonically until the limit of the precision of this calculator was reached. He, who suspects now that the sequence starting with 5 is monotonically increasing, is right: in fact, *any* sequence generated with the function f of this example, is monotonically increasing as soon as *two consecutive terms* are both $\not\equiv 0 \pmod 3$; all other sequences tend to the limit 1 (see [48]). □

EXAMPLE 2. f(n) := n' − n", where n' is the integer formed by arranging the decimal digits of n in *descending* order and n" is obtained by arranging them in *ascending* order.

Starting with 1983, the following iterative sequence shows up: 1983, 8442, 5994, 5355, 1998, 8082, 8532, 6174, 6174. Other example: 1984, 8352, 6174. D.R. KAPREKAR (see [28] and [29]) observed that any starting number with *four* decimal digits, not all equal, runs into 6174 within 8 steps. This number is now known as "Kaprekar's Constant". It is not difficult to see that any sequence generated with this f remains *bounded*. □

EXAMPLE 3. $f(n) := \sigma(n) - n$, where σ is the function denoting the sum of the divisors of n.

Here, f(n) is usually called the sum of the *aliquot* divisors of n; the f-iterative sequence starting with n is called the *aliquot sequence* of n. If $n = p \cdot q$, where p and q are distinct prime numbers, then we have $f(n) = 1 + p + q$. Consider, e.g., the number $9 = 1 + 3 + 5$, then $f(3 \cdot 5) = f(15) = 9$. Since $15 = 1 + 3 + 11$, we have $f(3 \cdot 11) = f(33) = 15$. Tracing back in this way, we find, e.g.,
$9 \leftarrow 3 \cdot 5 = 15 \leftarrow 3 \cdot 11 = 33 \leftarrow 3 \cdot 29 = 87 \leftarrow 3 \cdot 83 = 249 \leftarrow 7 \cdot 241 = 1687 \leftarrow 17 \cdot 1669 = 28373$, and so on. In this way, we are constructing *monotonically decreasing* aliquot sequences. The existence of such sequences of *any* prescribed length easily follows from the truth of the (somewhat sharpened) Goldbach Conjecture, which says that every even number > 2 is the sum of two *different* primes. □

2. GENERAL RESULTS

We consider the f-iterative sequence

$$n_0 = n,\ n_1 = f(n_0),\ \ldots,\ n_i = f(n_{i-1}) = f^i(n_0),\ \ldots \quad (2.1)$$

This sequence is called *periodic* when there are integers $i_0 \geq 0$ and $\ell \geq 1$ such that, for all $i \geq i_0$, $n_{i+\ell} = n_i$. The set $\{n_{i_0}, n_{i_0+1}, \ldots, n_{i_0+\ell-1}\}$ is called a *cycle* of f and ℓ the *length* of the cycle.

A *function* $f : \mathbb{N} \to \mathbb{N}$ is called periodic if (2.1) is periodic for all initial values $n \in \mathbb{N}$ and if f has only *finitely* many cycles. The following theorem gives a necessary and sufficient criterion for periodicity.

THEOREM 2.1. ([5]). *The function* $f : \mathbb{N} \to \mathbb{N}$ *is periodic if and only if there exists an* $N \in \mathbb{N}$ *and for all* $n > N$ *a* $k_0 = k_0(n)$ *such that for all* $n > N$

$$(2.2) \qquad f^{k_0}(n) < n.$$

A function $f : \mathbb{N} \to \mathbb{N}$ is called *uniformly periodic* if for all $n > N$ there exists a $k_0 \geq 1$, *independent* of n, such that $f^{k_0}(n) < n$. Many known periodic functions are uniformly periodic, but it is not difficult to construct non-uniformly periodic functions (cf. [5]).

There are only a few papers which give general results on f-iterative sequences. STEWART [52] made a thorough study of sequences generated by f defined by

$$(2.3) \qquad f(n) := \sum_{i=0}^{k} P(a_i),$$

where n has the unique representation to a given base B: $n = \Sigma_{i=0}^{k} a_i B^i$ $(a_k \geq 1)$, and where P is an arbitrary function mapping $\{0,1,\ldots,B-1\}$ into $\mathbb{N}$. He showed that this f is uniformly periodic and he gave an algorithm for the *evaluation* of the number N which has the property that $f(N) \geq N$ and $f(n) < n$ for all $n > N$. This N is of crucial importance for the determination of all cycles of the periodic function f. KULLMAN [32] called Stewart's paper the definitive theoretical work for the class of functions defined in (2.3).

BURKARD [5] studied periodic and uniformly periodic functions without mentioning Stewart's work. Cf. also HINTZ [24].

How can we determine cycles of an iterative sequence? Usually, a scientific pocket calculator has not much memory, so that *storage* of computed terms and comparison of a newly computed term with all previous terms is impossible when large cycles are involved. Fortunately, a better method for detecting cycles is known (although in all the papers on iterative sequences to which we refer, this method has not been mentioned explicitly): start with $n_0 = n$ and, for $i = 1,2,\ldots,$ compute, simultaneously,

$n_i = f(n_{i-1})$ and $n_{2i} = f(f(n_{2i-2}))$, and compare n_i and n_{2i}. Only the three integers i, n_i, and n_{2i} have to be stored. If for some $i = i_0$ we find $n_{i_0} = n_{2i_0}$, then we have detected a cycle, the length of which is a divisor of i_0. Starting now with n_{i_0}, the cycle and its length can be found by computing $n_{i_0+1}, n_{i_0+2}, \ldots$, and comparing each term with n_{i_0}. KNUTH [31] attributes this cycle detecting method to Floyd. For more about this problem, see SEDGEWICK & SZYMANSKI [49]. An interesting application of this method to the problem of factorization was devised by POLLARD [40].

3. A CATALOGUE OF f-ITERATIVE SEQUENCES

We present here a catalogue of functions f, iterative sequences of which have been studied in the literature. Of course, this list is by no means complete, although in our opinion the most important functions have been included. Also, our list of references is not complete. Those marked by an asterisk (*) contain a set of valuable additional references.[1)]

For every (class of) function(s) we shall indicate whether or not it is known to be (uniformly) periodic and, if not, whether (arbitrarily long) monotonically increasing or decreasing sequences exist.

We classify the functions in three classes, namely

I f(n) is some function of the digits of n in some base B representation system;

II f(n) is a function of certain divisors of n;

III miscellany.

I f(n) is a function of the B-adic (k+1)-digit number n,

where $n = \sum_{i=0}^{k} a_i B^i$, $0 \le a_i \le B-1$ $(0 \le i \le k-1)$, $1 \le a_k \le B-1$.

I.1 $f(n) = \sum_{i=0}^{k} P(a_i)$, where P maps the set $\{0,1,\ldots,B-1\}$ into $\mathbb{N}$. STEWART [52] showed that there exists an integer $N = N(P)$ such that $f(N) \ge N$ and such that $f(n) < n$ for every $n > N$. This

1) The author would be grateful to anyone who sends him other additional material.

implies that all f-iterative sequences of the class I.1 are uniformly periodic. For special choices of the function P this result has been rediscovered by many writers after Stewart.

I.1.a $P(a) = a^t$, $t \in \mathbb{N}$, t fixed.

For t = 1 the cycles are the sets $\{i\}$, $1 \le i \le B-1$.
For B = 10, all cycles were computed by PORGES [42] for t = 2, ISÉKI [25] for t = 3, CHIKAWA, ISÉKI & KUSAKABE [8] for t = 4, CHIKAWA, ISÉKI, KUSUKABE & SHIBAMURA [9] for t = 5, AVANESOV & GUSEV [1] for t = 6 and t = 7, TAKADA [53] for t = 8, and, finally, ISÉKI & TAKADA [26] for t = 9.
YOSHIGAHARA [59] obtained all cycles for B = 100 and t = 2.
For general base B, HASSE & PRICHETT [22] investigated the cycles in the case t = 2. For $2 \le B \le 10$, DEIMEL, Jr. & JONES [13] listed all cycles of length 1 for which t = k + 1.

I.1.b $P(a) = a!$

For B = 10, POOLE [41] tracked down all cycles of length 1, viz., {1},{2}, {145} and {40585}, and KISS [30] found all other cycles (viz., two of length 2 and one of length 3).

I.1.c $P(a) = \binom{B}{a}$

For B = 10, KISS [30] found all cycles (one of length 1 and one of length 2).

I.1.d $P(a) = a!\cdot\binom{B}{a}$

For B = 10, KISS [30] found all cycles (one each of length 2,4 26 and 39).

I.2 $f(n) = \prod_{i=0}^{k} P(a_i)$, where P maps the set $\{0,1,\ldots,B-1\}$ into $\mathbb{N} \cup \{0\}$.

STEWART [52] showed that there exists an integer N > 0 for which $f(N) \ge N$ and $f(n) < n$ for every $n > N$, only in the case that $1 \le M' \le M < B$ and $P(a) \ge a$ for at least one $a \in \{0,1,\ldots,B-1\}$. Here, $M' = \max_{1 \le a' \le B-1} P(a')$ and $M = \max(P(0),M')$. Moreover, he showed that in all other cases either N = 0 or N does not exist.

I.2.a $P(a) = a + t$, $t \in \mathbb{N} \cup \{0\}$, t fixed.

For $B = 10$ WAGSTAFF [57] made a thorough computational study of the ten cases $0 \le t \le 9$. (For $t \ge 10$ we have $f(n) > n$ for all $n \in \mathbb{N}$.) He showed that for $t = 1$ and $n > 18$, either $f(n) < n$ or $f(f(n)) < n$, hence in this case f is uniformly periodic. For $2 \le t \le 6$ he gave numerical evidence and a heuristic argument that every sequence remains bounded (and hence is periodic), whereas for $t \ge 7$ virtually every sequence tends to infinity.

I.3 $f(n) = (\sum_{i=0}^{k} a_i)^t$, $t \in \mathbb{N}$, t fixed.

It is not difficult to modify Stewart's proof, mentioned in I.1, in order to show that for any fixed B and t this function f is uniformly periodic. MOHANTY & KUMAR [39] found all cycles for $B = 10$, $2 \le t \le 10$.

I.4 $f(n) = n' - n''$ where n' is the integer formed by arranging the base B digits of n in *descending* order and n'' by arranging them in *ascending* order.

In the definition of f the following convention is generally adopted: if n is a k - digit number and f(n) has k-1 digits, then a leading zero is added to f(n). Example:
$(B=10)$ $n = 1121$, $f(n) = 2111 - 1112 = \underline{0}999$, $f(f(n)) = 9990 - 0999 = 8991$. When we neglect numbers all whose digits are equal, this means that iterating f on k-digit numbers always leads to k-digit numbers.

This function is *not* periodic because the number of cycles is not finite. KAPREKAR [28,29] was the first one to discover that when we start with a four-digit number (base 10), not all digits being identical, then within 8 steps the cycle {6174} is reached.
JORDAN [27] studied cycles of length 1 for any base B.
For B = 2,3,..., 12, TRIGG listed all cycles for two-digit numbers in [54] and all those for five-digit numbers in [55].
HASSE [21] presented a detailed study of the problem for two-digit numbers, for general base B.
HASSE & PRICHETT [23] studied cycles of length 1 for four-digit,

general base B integers. They showed that only for $B = 2^n5$ with $n = 0$ or n odd, there exists a four-digit integer such that *every* f-iterative sequence tends to this integer. Moreover, this integer is explicitly given as

$$dB^3 + (d'-1)B^2 + (B-d'-1)B + (B-d), \text{ where } (d,d') = (2^n3, 2^n).$$

PRICHETT [43] gave a complete characterization of all cycles in the case of five-digit B-adic integers.

LAPENTA, LUDINGTON & PRICHETT [33] presented an algorithm for the calculation of cycles of length 1 in the case of r-digit, B-adic integers, for all $r \geq 1$ and all $B \geq 2$.

LUDINGTON [35] showed that for any fixed base B there exists an $r_0 \in \mathbb{N}$ such that for all r-digit numbers with $r > r_0$ there is *no* cycle of length 1.

PRICHETT , LUDINGTON & LAPENTA [44] showed that for B = 10 the only cycles of length 1 are 495 (for 3-digit numbers) and 6174 (for 4-digit numbers).

<u>I.5</u> $f(n) = (n+t)^*$, $t \in \mathbb{N}$, t fixed, where m^* is the number obtained by reversing the digits of m.

According to SIERPIŃSKI [50], Rokowska and Schinzel showed that for B = 10 and t = 5 all f-iterative sequences are periodic. However, the function f is *not* periodic, since the number of cycles is infinite, as was shown by Gorzkowski by the following result:

The f-iterative sequence starting with $n_0 = 10^{2k+3} + 10^{k+1} + 1$ *is periodic with period length* $36 \cdot 10^k$ (for any fixed $k \in \mathbb{N} \cup \{0\}$).

SIERPIŃSKI [50] also gave results for other values of t (with B = 10), e.g., f is periodic for t = 1, 2, 3, 4, 7, 8, 9 and 11. For t = 10, f is *not* periodic, for t = 6 the question is still open.

<u>II</u> f(n) is a function of certain divisors of n.

<u>II.1</u> $f(n) = \sigma(n) - n$, where $\sigma(n)$ denotes the sum of the divisors of n. f(n) is usually called the sum of the *aliquot* divisors of n.

Cycles of length 1 are known as *perfect* numbers, cycles of length 2 as *amicable* pairs. It is very plausible that this f is *not* periodic, although a proof may be difficult. H.W. Lenstra, Jr. showed that monotonically increasing aliquot sequences of any prescribed length do exist.

Much statistical material has been collected. For odd n_0, aliquot sequences usually tend very quickly to 1. For even n_0, some sequences tend to 1, whereas many more become so large that computation of subsequent terms is very difficult (since factorization is required).

GUY & SELFRIDGE (cf. [18]) have conjectured that almost all even aliquot sequences tend to infinity. For references and related problems on cycles, consult GUY [20], in particular problems B1, B4, B6 and B7.

Very recently, the present author [47] has collected new numerical evidence for the existence of infinitely many cycles of length 2. Moreover, he has found new very large cycles of length 2, the largest pair consisting of two 282-digit numbers ([46]).

II.2 $f(n) = \sigma^*(n) - n$, where $\sigma^*(n)$ denotes the sum of the *unitary* divisors of n (a divisor d of n is unitary if $\gcd(d,n/d) = 1$).

It is not known whether this f is periodic or not. It is known that monotonically increasing unitary aliquot sequences of any prescribed length do exist (cf. [45]). For references and pertinent remarks concerning the possible existence of unbounded unitary aliquot sequences, consult GUY [20], problems B3 and B8.

II.3 $f(n) = g(n) - n$, where $g(n)$ is a multiplicative function of n. In many cases, this f can be interpreted as the sum of *certain* divisors of n, and it includes, as special cases, the examples II.1 and II.2.

In his thesis, the present author ([45]) made a detailed study of iterative sequences generated by this f. For some specific choices of g he showed the existence of *unbounded* sequences. For example, if g(n) is the multiplicative function defined by $g(p^e) := p^e + p^{e-1}$, for prime p and integer $e \geq 1$, then for the sequence starting with $n_0 = 318 = 2\cdot3\cdot53$, we have

$$f^{19j+12}(n_0) = 9870\cdot 27^j = 2\cdot 3^{3j+1}5\cdot 7\cdot 47, \; j = 0,1,\ldots,$$

and moreover, it increases *monotonically*.

II.4 $f(n) = \sigma(n) - n \pm 1$.

It is not known whether these two f are periodic or not. For not too large initial values all f-iterative sequences are known to be bounded. See GUY [20], problem B5 for some references.

II.5 $$f(n) = \begin{cases} \sigma(n/2) & \text{if } n \text{ is even} \\ \sigma(n) & \text{if } n \text{ is odd} \end{cases}.$$

This function was recently studied by BEDOCCHI [3]. Its f-iterative sequences are much easier to compute than, e.g., aliquot sequences, because, roughly spoken, *large* prime divisors of n tend to be transformed by f into *smaller* prime divisors of f(n). Some experiments, carried out by the present author, indicate that most sequences tend very fast to infinity.

II.6 $n = \prod_{i=1}^{k} p_i$, $p_1 \le p_2 \le \ldots \le p_k$, p_i primes.

II.6.a $f(n) = \sum_{i=0}^{k} p_i p_{k-i}$, $p_0 := 1$.

GRUŻEWSKI & SCHINZEL [17] proved that this f is uniformly periodic and the cycles are {16}, {18}, {35,39} and {22,26,30}.

II.6.b $f(n) = d + \sum_{i=1}^{k} p_i$, $d \in \mathbb{N}$, d fixed.

BURKARD [5] proved that this f is uniformly periodic for any fixed $d \in \mathbb{N}$ and he gave all cycles for $1 \le d \le 5$.

II.7 $n = \prod_{i=1}^{k} p_i^{a_i}$, unique factorization of n into primes p_i.

II.7.a $f(n) = d + \prod_{i=1}^{k} p_i$, $d \in \mathbb{N}$, d fixed.

BURKARD [5] proved that also this f is uniformly periodic for any fixed $d \in \mathbb{N}$.

II.7.b $f(n) = 1 + \sum_{i=1}^{k} a_i p_i$.

This function is uniformly periodic, and was studied, among others, by CADOGAN & CALLENDER [7] and by BELLAMY & CADOGAN [4].

III MISCELLANY.

III.1 $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$.

This f is notorious. There is overwhelming numerical evidence (partly because f is so easy to compute) that it is uniformly periodic, but no proof of this is known. Many references are given in GUY [19]. Also see [12] and [16].

III.2 $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (qn+1)/2 & \text{if } n \text{ is odd} \end{cases}$, $q \in \mathbb{N}$, q odd, q fixed.

This is a generalization of the previous example, studied by STEINER [51]. Also see [11].

III.3 $f(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3 \\ [n\sqrt{3}] & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$.

The present author has shown ([48]) that as soon as two consecutive terms in a sequence generated by this f are $\not\equiv 0 \pmod 3$, then this sequence tends monotonically to infinity. He has found that only 459 of the f-iterative sequences with initial value ≤ 100000 tend to 1, all others tend to infinity. He conjectures that almost all sequences tend to infinity.

III.4 $\vec{n} = (n_1, n_2, \ldots n_k)$, $k \geq 2$, k fixed, $n_i \in \mathbb{N}$,
$\vec{f}(\vec{n}) = (|n_1-n_2|, |n_2-n_3|, \ldots, |n_{k-1}-n_k|, |n_k-n_1|)$.

This is an example of a vector *function* $\vec{f}$ which has received much attention in the literature, in particular the case $k = 4$. It is not difficult to see that this $\vec{f}$ is periodic in the sense that for any given initial vector $\vec{n}_0$ its $\vec{f}$-iterative sequence tends to a cycle of vectors (possibly the null vector). CIAMBERLINI & MARENGONI [10] showed that if k is a power of 2 then any $\vec{f}$-iterative sequence leads to the null vector

$(0,0,\ldots,0)$. FREEDMAN [15] published a study which is still worth reading. He showed that if k is *not* a power of 2 then $\vec{f}$-iteration does not generally yield the null vector. Moreover, he proved that for any $k > 2$ and any index i, there is an $\vec{f}$-iterative sequence with i terms before a cycle occurs.
An interesting generalization of this problem to vectors of (four) *real* numbers was described by LOTAN [34]. He showed that the $\vec{f}$-iterative sequence starting with any vector $\vec{n}_0$ of four *real* numbers leads within a *finite* number of steps to the null vector, except only when $\vec{f}(\vec{n}_0)$ is of the form $(1,q,q^2,q^3)$, where q is the positive solution of the equation $q^3 - q^2 - q - 1 = 0$ (or of a form derived from $(1, q, q^2, q^3)$ by trivial transformations).
See also the papers by BURMEISTER, FORCADE & JACOBS [6], MILLER [37], ZVENGROWSKI [60], DUMONT & MEEUS [14] and WEBB [58].
LUDINGTON FURNO [36] showed that for every k there are only a finite number of cycles (except for constant multiples). Moreover, she explicitly determined the vectors which belong to the various cycles.

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